

**U.S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE
NATIONAL METEOROLOGICAL CENTER**

OFFICE NOTE 9

**PRELIMINARY SURVEY IN TRUNCATION ERRORS IN THE
NUMERICAL SOLUTION OF PARTIAL DIFFERENT EQUATIONS**

G. Cressman

November 1958

**This is an unreviewed manuscript, primarily intended for informal
exchange of information among NMC staff members**

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Preliminary Survey in Truncation Errors
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The past decade was marked by a burst of interest and activity which has animated the field of numerical methods in general and the numerical solution of differential equations in particular. The interest in numerical analysis continues to swell, primarily because of the success of the large scale calculating machines. Progress in mechanical computation has opened up whole areas heretofore deemed inaccessible. The time required for computation in a given instance is reduced, which, in turn, has thrown us into lengthy analytical studies and placed additional emphasis on numerical methods.

We concede the necessity to be well grounded in both the principles and the existent techniques of numerical analysis in order to adapt best available methods to the machine at our disposal, to comprehend the inherent limitations of our methods, and to devise error controls and modified techniques which are best applicable to that machine. We want to estimate effectively the reliability of any numerical result we obtain, improve that reliability, whenever it is possible, and control that reliability in that improved state.

Whereas the availability of large scale rapid calculations has made feasible the numerical solution of many problems of previously prohibitive complexity, the effective use of such devices depends strongly upon continued advancement of research in relevant fields of mathematical analysis.

One of the most powerful methods for solving a partial differential equation numerically is the method of finite differences, in which one first approximates the differential equation by a difference equation and then solves this resulting difference equation. We may suspect that since we fail to approach a limit in obtaining numerical results by these replacements, that these new machines in using simplest possible methods for the solution of our partial differential equations will attain desired accuracy by means of many very short steps or by an ultrafine mesh. But for bigger and bigger problems, the danger of accumulated error due to many steps and the limitations on fineness of mesh imposed by limited high speed memory capacity will oppose obtaining accuracy by extravagantly short steps and ultrafine mesh.

For all these reasons our new machines intensify our search for methods simple enough to be coded for practical use but powerful enough to produce adequate accuracy without unreasonable reduction in step length or mesh size. And also, it is often convenient to retain present methods boosted and improved with effective error analysis.

Often in situations representing real problems, a knowledge of the physics of the system will give a clear indication of any marked inaccuracy in the solution. In the absence of such guidance, however, the best that can be done is a check on the solution by means of a more accurate approximation to the equation using differences of higher order than those used to obtain the solution originally. It cannot be too strongly remarked, however, that such higher difference formulae should not be used in the original calculation since they are equivalent to replacing the original differential equation by one of higher order, and may thus introduce spurious detail into the solution. A less dangerous procedure is to use higher difference formulae to estimate the error of a result obtained by the use of the simpler methods.

A better method is to decrease the size of the interval keeping in mind the possibilities of the storage capacity of our machine. Even this does not necessarily yield results of increasing accuracy. We are soon to see that associated with a numerical method is an inherent error by which we are bound as long as we employ that method. It is important that we engaged in applications of finite differences, keep these limitations in mind, lest we find ourselves incurring heavy labor, in vain, aiming at an accuracy that is not, in the first place, an inherent concomitant of the process being employed.

While Courant, Friedrichs and Lewy studied the parabolic equation representing the one-dimensional flow of heat in a conducting wire, or the diffusion of a liquid or gas along a porous tube, from the finite difference point of view, they were lead to their fundamental theorem on the relationship between the space and time intervals. $(c^2 \frac{\partial^2 \omega}{\partial x^2} = \frac{\partial \omega}{\partial t})$

The important contribution of Courant and his co-workers was to show that it is not possible to choose the step in x and the step in t arbitrarily if a stable solution is to be obtained. By considering the difference between the solution of the differential equation and that of the difference equation, they showed that the error was bounded only if $\gamma = \frac{c^2 \delta t}{(\delta x)^2} \leq \frac{1}{2}$

and that it grows exponentially with t when $\gamma > \frac{1}{2}$. This implies that when solving an equation of that general type steps in t and steps in X must be chosen so as to make $\frac{c^2 \delta t}{(\delta x)^2} \leq \frac{1}{2}$

and that, in consequence, unlimited decrease in the steps in X will not lead to improved accuracy unless accompanied by a suitable decrease in steps in t . So that when the size of the interval is decreased in any direction, care must be taken to decrease other intervals appropriately. Unfortunately, more general versions of Courant's results are not always available but the known forms may give some idea of the dimensions involved.

If your intuition has been appealed to in this discourse, it has not been totally relied upon. Assertions (if the statements contained herein can be so dignified) were not backed up with practical examples, but problems of error analysis have been treated as adequately as time and preparation permitted.

So, you are looking for a panacea? Beware - it will never suffice every ill of any numerical method. You already know the answers? Even though it is suspected that you mean mere approximations to the answers, with your keen physical insight into your very own problem and past experience with the special behavior of your previous results, what you already know is probably very good; and there is a high probability that that can be improved and controlled in that improved state.

A "Marching" problem was defined by Richardson as one in which the integral can be stepped out from a part of the boundary. The prediction of astronomical events belongs to the marching class. Also, weather prediction belongs to the same, at least if you believe that future weather is determined by present weather together with astronomical events which are foreknown.

Consider
$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial (\psi, \nabla^2 \psi + f)}{\partial (x, y)} = 0,$$

our barotropic non-divergent equation, for which we will be given initially a field of ψ 's used to extrapolate into the future using short time steps. Our equation in expanded form reads

$$\frac{\partial \nabla^2 \psi}{\partial t} = -\psi_x (\nabla^2 \psi + f)_y + \psi_y (\nabla^2 \psi + f)_x.$$

No investigation was made herein as to the relative magnitudes of the steps considered in x , y , and t . We shall assume that some kind of rapport has been reached and for a given step in t , corresponding steps in x and y are likewise determined. It appears that if we knew any solution that if we also got a series of numerical solutions we could note any growth of instability and set our bounds.

It is very important that the differences should be "centered". As one would suspect, our approximations are less accurate near the ends of our interval than near the middle. Central difference formulae have usually proved superior converging more rapidly than other formulae and having smaller remainder terms.

From our field of ψ 's we estimate our ψ_x and ψ_y according to the rule:

$$\psi_x = \frac{\psi(x+h, y, t) - \psi(x, y, t)}{h}$$

$$\psi_y = \frac{\psi(x, y+k, t) - \psi(x, y, t)}{h}$$

These give forward first-difference quotients. The backward first difference quotients are

$$\psi_{\bar{x}} = \frac{\psi(x, y, t) - \psi(x-h, y, t)}{h}$$

$$\frac{\psi_{\bar{y}}}{k} = \frac{\psi(x, y, t) - \psi(x, y-k, t)}{k},$$

whereas the first central difference

$$\frac{\psi_x = \psi(x + \frac{h}{2}, y, t) - \psi(x - \frac{h}{2}, y, t)}{h}, \text{ and analogously}$$

for ψ_y . However, we have no continuous data, but it is assigned at discrete points which leaves us no alternative but to work with what we have, lest we start approximating answers obtained from approximated data which might be worst than backward and forward differences. For points at a left end we might use a forward difference; for points at a right end we might use a backward difference.

What errors have we made in this estimating our first partials? Scarborough states that the inherent (truncation) error in the difference-equation solution of a differential equation can be found by expressing the difference quotients in terms of derivatives and this can be done by means of Taylor's formulae.

In the above formulae "t" does not vary so we consider ψ a function of x, y:

$$\psi(x+h, y) = \psi(x, y) + h\psi_x + \frac{h^2}{2!} \psi_{xx} + \frac{h^3}{3!} \psi_{xxx} + \frac{h^4}{4!} \psi_{xxx} + \dots$$

$$\psi(x-h, y) = \psi(x, y) - h\psi_x + \frac{h^2}{2!} \psi_{xx} - \frac{h^3}{3!} \psi_{xxx} + \frac{h^4}{4!} \psi_{xxx} - \dots$$

Forming the forward first difference quotient we see that

$$\frac{\psi(x+h, y) - \psi(x, y)}{h} = \psi_x + \frac{h}{2} \psi_{xx} + \frac{h^2}{3!} \psi_{xxx} + \text{higher order terms.}$$

We overestimated ψ_x by $\frac{h}{2} \psi_{xx} + \frac{h^2}{3!} \psi_{xxx} + \text{higher order terms}$, assuming ψ possesses derivatives of all orders. Similarly

$$\frac{\psi(x, y) - \psi(x-h, y)}{h} = \psi_x - \frac{h}{2} \psi_{xx} + \frac{h^2}{3!} \psi_{xxx} - \dots \text{higher order terms.}$$

We underestimated ψ_x by $\frac{h}{2} \psi_{xx} - \frac{h^2}{3!} \psi_{xxx} + \dots$ higher order terms. This leads us to feel that even

$$\frac{\psi(x+h, y) - \psi(x-h, y)}{2h} = \psi_x + \frac{h^2}{3!} \psi_{xxx} + \dots \text{higher order terms in}$$

$\frac{h^4}{5!}, \frac{h^6}{7!}$ would have yielded better results, being a "centered" formula and taken at points we are given.

We obtain next $\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$.

And since we have no material for its estimate at the boundary of our region we will assign $\nabla^2 \psi$ a neutral value of zero there.

It is customary to use only the first approximation to the Laplacian obtained by neglecting the fourth and all higher differences, where also we let $h = k$. We define

$$\psi_{\bar{x}x} = \frac{\psi_x - \psi_{\bar{x}}}{h} = \frac{\psi(x+h, y) - 2\psi(x, y) + \psi(x-h, y)}{h^2}$$

We define

$$\psi_{\bar{y}y} = \frac{\psi_y - \psi_{\bar{y}}}{h} = \frac{\psi(x, y+h) - 2\psi(x, y) + \psi(x, y-h)}{h^2}$$

Then

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \approx \psi_{\bar{x}x} + \psi_{\bar{y}y} =$$

$$\frac{1}{h^2} [\psi(x+h, y) + \psi(x-h, y) + \psi(x, y+h) + \psi(x, y-h) - 4\psi(x, y)]$$

Again we inquire about the error we are making. Scarborough defines $\psi_{\bar{x}x}$ with the help of Taylor's expansions as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2h^2}{4!} \frac{\partial^4 \psi}{\partial x^4} + \text{terms in } h^4, h^6, \text{ etc.}$$

Likewise

$$\psi_{\bar{y}y} = \frac{\partial^2 \psi}{\partial y^2} + \frac{2h^2}{4!} \frac{\partial^4 \psi}{\partial y^4} + \text{terms in } h^4, h^6, \text{ etc.}$$

Adding

$$\psi_{\bar{x}x} + \psi_{\bar{y}y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{2h^2}{4!} \left(\frac{\partial^4 \psi}{\partial x^4} + \frac{\partial^4 \psi}{\partial y^4} \right) + \text{terms in } h^4, h^6, \text{ etc.}$$

The error thus committed in writing

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \psi_{\bar{x}x} + \psi_{\bar{y}y}$$

is thus a power series in even powers of h , the principal part of the error being the first term of the series and equal to

$$\frac{2h^2}{4!} \left(\frac{\partial^4 \psi}{\partial x^4} + \frac{\partial^4 \psi}{\partial y^4} \right)$$

There are other formulae for the Laplacian with their corresponding error estimates, a few others of which it may do well to mention here.

This formula is due to Kunz presented in his 1957 edition of Numerical Analysis. Suppose the Laplacian is desired at position (x_0, y_0) . We define U and v by the equations

$$U = \frac{x - x_0}{h}, \quad v = \frac{y - y_0}{k}, \quad h = k \text{ possibly.}$$

By Stirling's interpolation formula he defines

$$\begin{aligned} \psi(x, y_0) = & \psi(x_0, y_0) + u \mu \delta_x \psi(x_0, y_0) + \frac{1}{2!} u^2 \delta_x^2 \psi(x_0, y_0) + \frac{1}{3!} u(u^2 - 1) \mu \delta_x^3 \psi(x_0, y_0) \\ & + \frac{1}{4!} u^2(u^2 - 1) \delta_x^4 \psi(x_0, y_0) + \frac{1}{5!} u(u^2 - 1)(u^2 - 4) \mu \delta_x^5 \psi(x_0, y_0) + \\ & \frac{1}{6!} u^2(u^2 - 1)(u^2 - 4) \delta_x^6 \psi(x_0, y_0) + \dots + \\ & \frac{1}{(2K-1)!} u(u^2 - 1)(u^2 - 4) \dots [u^2 - (K-1)^2] \mu \delta_x^{2K-1} \psi(x_0, y_0) + \\ & \frac{1}{2K!} u^2(u^2 - 1)(u^2 - 4) \dots [u^2 - (K-1)^2] \delta_x^{2K} \psi(x_0, y_0) + \dots \end{aligned}$$

where

$$\mu \delta_x^{2k-1} \psi(x_0, y_0) = \frac{1}{2} \left[\delta_x^{2k-1} \psi(x_0 - \frac{h}{2}, y_0) + \delta_x^{2k-1} \psi(x_0 + \frac{h}{2}, y_0) \right],$$

a mean difference. By definition of u and v ,

$$\frac{\partial^2 \psi(x, y_0)}{\partial x^2} \bigg|_{x=x_0} = \frac{1}{h^2} \frac{\partial^2 \psi(x, y_0)}{\partial u^2} \bigg|_{u=0}$$

$$\frac{1}{h^2} \left[\delta_x^2 \psi(x_0, y_0) - \frac{1}{12} \delta_x^4 \psi(x_0, y_0) + \frac{1}{90} \delta_x^6 \psi(x_0, y_0) + \dots \right],$$

where the subscript x on the δ_x indicates that the differences are formed with respect to x . In particular

$$\delta_x^2 \psi(x_0, y_0) = \psi(x_0 - h, y_0) - 2\psi(x_0, y_0) + \psi(x_0 + h, y_0),$$

$$\delta_x^4 \psi(x_0, y_0) = \psi(x_0 - 2h, y_0) - 4\psi(x_0 - h, y_0) + 6\psi(x_0, y_0) - 4\psi(x_0 + h, y_0) + \psi(x_0 + 2h, y_0).$$

The δ_y^2 and δ_y^4 can be handled similarly yielding

$$\delta_y^2 \psi(x_0, y_0) = \psi(x_0, y_0 - h) - 2\psi(x_0, y_0) + \psi(x_0, y_0 + h),$$

$$\delta_y^4 \psi(x_0, y_0) = \psi(x_0, y_0 - 2h) - 4\psi(x_0, y_0 - h) + 6\psi(x_0, y_0) - 4\psi(x_0, y_0 + h) + \psi(x_0, y_0 + 2h).$$

Then

$$\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = \frac{1}{h^2} \left[\delta_x^2 \psi(x_0, y_0) - \frac{1}{12} \delta_x^4 \psi(x_0, y_0) + \frac{1}{90} \delta_x^6 \psi(x_0, y_0) + \dots \right]$$

$$+ \frac{1}{h^2} \left[\delta_y^2 \psi(x_0, y_0) - \frac{1}{12} \delta_y^4 \psi(x_0, y_0) + \frac{1}{90} \delta_y^6 \psi(x_0, y_0) + \dots \right]$$

For sake of completeness he deems it desirable to obtain the second-order approximation to the Laplacian by taking into account the fourth differences. Then with neglect of only sixth, eighth and higher differences

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \approx \nabla_h^2 \psi(x_0, y_0)$$

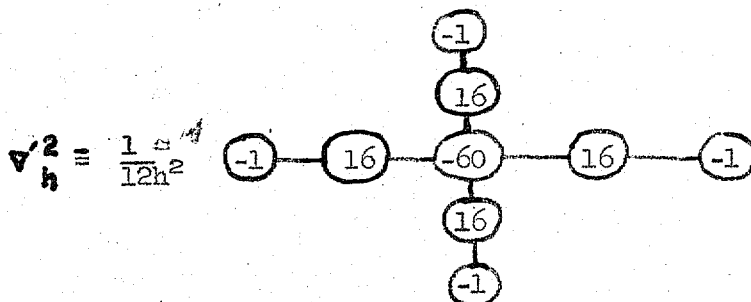
where

$$\nabla_h^2 \psi(x_0, y_0) = \frac{1}{12h^2} \left[\delta_x^2 \psi(x_0, y_0) - \frac{1}{12} \delta_x^4 \psi(x_0, y_0) + \delta_y^2 \psi(x_0, y_0) - \frac{1}{12} \delta_y^4 \psi(x_0, y_0) \right].$$

If we abbreviate $\psi(x_0 + nh, y_0) = \psi_{n0}$ and $\psi(x_0, y_0 + mh) = \psi_{0m}$, then

$$\nabla_h^2 \psi(x_0, y_0) = \frac{1}{12h^2} \left[-(\psi_{20} + \psi_{02} + \psi_{-20} + \psi_{0,-2}) + 16(\psi_{10} + \psi_{01} + \psi_{-10} + \psi_{0,-1}) - 60\psi_{00} \right].$$

For a quick and comprehensive visual aid we present what is called in the literature a stencil or lozenge below



The leading term truncated here is

$$\frac{1}{90h^2} \left(\delta_x^6 \psi(x_0, y_0) + \delta_y^6 \psi(x_0, y_0) \right).$$

Realizing that in most difference tables the differences of ordinarily behaved functions eventually level off and finally become null, in general, we suspect a better approximation with these included terms.

Since it is often good to compute by one method and check by another one other formula still will be presented. Booth and Milne, Booth using Milne as a reference, are chief advocates of the identity to follow below.

We define an operator H for approximating the Laplacian

$$H\psi = [\psi(x_0+h, y_0) + \psi(x_0-h, y_0) + \psi(x_0, y_0+h) + \psi(x_0, y_0-h) - 4\psi(x_0, y_0)]$$

and by stencil

$$\begin{array}{ccc} & 1 & \\ 1 & & -4 & 1 \\ & 1 & \end{array} = H.$$

And we define 2X likewise for approximating the Laplacian.

$$2X\psi = \psi(x_0-h, y_0+h) + \psi(x_0+h, y_0+h) + \psi(x_0-h, y_0-h) + \psi(x_0+h, y_0-h) - 4\psi(x_0, y_0)$$

and again by stencil or lozenge

$$\begin{array}{ccc} & 1 & \\ & & -4 & \\ & 1 & \end{array} = 2X.$$

Introduce operations E_x and E_y defined as follows:

$$E_x\psi(x, y) = \psi(x+h, y)$$

$$E_y\psi(x, y) = \psi(x, y+h), \text{ with symbolic associations}$$

$$E_x = \exp h \frac{\partial}{\partial x}$$

$$E_y = \exp h \frac{\partial}{\partial y}.$$

From the expression for H we have

$$H(\psi) = (E_x + E_x^{-1} + E_y + E_y^{-1} - 4)(\psi).$$

Employing an exponential expansion

$$H(\psi) = h^2 \nabla^2(\psi) + \frac{1}{12} h^4 \nabla_0^4 \psi + \frac{1}{360} h^6 \nabla_0^6 \psi + \dots$$

if we let $\nabla_0^n \psi = \frac{\partial^n \psi}{\partial x^n} + \frac{\partial^n \psi}{\partial y^n}$

Again we see that if we let $\nabla^2 \psi = \frac{1}{h^2} H(\psi)$, our error is in overestimating ∇^2 by $\frac{h^2}{12} \nabla_0^4 \psi$ by a first term approximation. And from $2X$, we have:

$$2X(\psi) = [E_x^{-1} E_y + E_x \cdot E_y + E_x^{-1} E_y^{-1} + E_x E_y^{-1} - 4](\psi),$$

which by employing again our exponential expansion leads to

$$2X(\psi) = 2h^2 \nabla^2 \psi + \frac{1}{6} h^4 \nabla_0^4 \psi + h^4 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{h^6}{180} \nabla_0^6 \psi + \frac{h^6}{12} \frac{\partial^4}{\partial x^2 \partial y^2} [(\nabla^2 \psi)] + \dots$$

whence

$$\nabla^2 \psi = \frac{1}{2h^2} [2X(\psi)] - O(h^2).$$

So, if we approximate $\nabla^2 \psi$ by $2X$ our error is in overestimating ∇^2 by $\frac{h^2}{12}$.

$(\nabla_0^4 \psi + 6 \frac{\partial^4 \psi}{\partial x^2 \partial y^2})$ by the first term approximation. So we notice that as is to

be expected from the effectively larger interval of differencing involved in calculating $2X$, the error term, although $O(h^2)$ = order of h^2 , has a larger coefficient than in the equation for $H(\psi)$.

In the special case of the Laplace equation where $\nabla^2 \psi = 0$

$$\frac{\partial^4 \psi}{\partial x^2 \partial y^2} = - \frac{\partial^4 \psi}{\partial x^4} = - \frac{\partial^4 \psi}{\partial y^4} = - \frac{1}{2} \nabla_0^4 (\psi).$$

Our equation for $2X(\psi)$ becomes

$$2X(\psi) = 2h^2 \nabla^2(\psi) - \frac{1}{3} h^4 \nabla_0^4(\psi) + \frac{h^6}{180} \nabla_0^6(\psi).$$

Combining this equation for $2X(\psi)$ with the equation for H we have

$$(4H + 2X)(\psi) = 6h^2 \nabla^2(\psi) + \frac{h^6}{60} \nabla_0^6(\psi) + \dots$$

or

$$\nabla^2 \psi = \frac{1}{6h^2} (4H + 2X)\psi - \frac{h^4}{360} \nabla_0^6 \psi + \dots$$

By stencil or lozenge representation ($4H+2X$) is

$$\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix}$$

=K according to Milne. This is still another

approximating formula for the Laplacian.

$$\nabla^2 \psi = \frac{K(\psi)}{6h^2} - \frac{h^4}{360} \nabla_0^6 (\psi) + \dots$$

$\nabla^2 \psi \approx \frac{K(\psi)}{6h^2}$ where $-\frac{h^4}{360} \nabla_0^6 (\psi)$ is the first term truncated. He also defines $N^2 = 2X - 2H =$

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \frac{h^4}{\partial x^2 \partial y^2} + O(h^6).$$

Milne replaces X and H by using K and N^2 and concludes that below is a true identity:

$$\nabla^2 = \frac{1}{6h^2} \left(K - \frac{K^2}{72} + \frac{K^3}{3240} - \frac{KN^2}{180} - \frac{K^4}{120,960} + \frac{K^2 N^2}{3780} - \frac{N^4}{504} + \dots \right)$$

when truncated to N terms, when applied to any polynomial in x and y of degree less than $2N+2$.

We had found our first partials in ψ . Our first partials for $(\nabla^2 \psi + f)$ are found in an analogous way. In finding our Jacobian, we are multiplying together approximated partial derivatives. It can easily be shown that the relative error in either product is the sum of the relative errors of the two factors multiplied, where we define a relative error as the actual error divided by the magnitude of the number in question. But we can expect the significant error terms in our time derivative to be the sum;

$$-(\text{error in } \psi_x)(\nabla^2 \psi + f)_y - [\text{error in } (\nabla^2 \psi + f)_y] \cdot \psi_x + (\text{error in } \psi_y) \cdot (\nabla^2 \psi + f)_x + [\text{error in } (\nabla^2 \psi + f)_x] \psi_y.$$

We assume here that we have $\nabla^2 \psi$ defined for every internal point with the value zero assigned for every point on the boundary. We proceed to obtain ψ . Our problem seems comparable to a Poisson equation in two dimensions.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -4\pi\rho(x,y) = f(x,y).$$

Milne has this to say: We overlay our region concerned with a square lattice having mesh length h and replace the original equation by an appropriate difference equation. The simplest such equation is $H(\psi) = h^2 f(x,y)$, which follows readily from our previous definition of H , the local error being

$$\frac{h^4}{12} \nabla_0^4 \psi, \text{ as previously stated, by a first approximation.}$$

We secure a somewhat more accurate difference equation by the formula for ∇^2 above, which gives

$$K\psi - \frac{K^2 \psi}{72} = 6h^2 f(x,y) + O(h^6).$$

For computational purposes this equation can be considerably improved if the function $f(x,y)$ is such that $\nabla^2 f(x,y)$ exists every where in the region. For in that case, we have approximately

$$K^2 \psi = 36h^4 \nabla^4 \psi = 36h^4 \nabla^2 f(x,y)$$

(since our operator

$$K = 6h^2 \nabla^2 + O(h^4))$$

and we replace the equation above with

$$K\psi = 6h^2 f(x,y) + \frac{h^4}{2} \nabla^2 f(x,y),$$

and our error has for its leading term $\left[\frac{K^3}{3240} - \frac{KN^2}{180} \right] \psi$, order of h^6 . If $f(x,y)$ happens to be harmonic, we get a simple equation

$$K\psi = 6h^2 f(x,y).$$

Should we use the equation

$$K(\psi) = 6h^2 f(x,y) + \frac{h^4}{2} \nabla^2 f(x,y)$$

We can use the familiar finite difference form

$$(4H+2X) \psi = K(\psi) =$$

$$4\psi(x+h,y) + 4\psi(x-h,y) + 4\psi(x,y+h) + 4\psi(x,y-h) + \psi(x+h,y+h) +$$

$$\psi(x-h,y+h) + \psi(x-h,y-h) + \psi(x+h,y-h) - 20\psi(x,y) =$$

$$6h^2 f(x,y) + \frac{h^4}{2} \nabla^2 f(x,y),$$

and from this

$$\psi(x,y) = \frac{1}{20} \left[4\psi(x+h,y) + 4\psi(x-h,y) + 4\psi(x,y+h) + 4\psi(x,y-h) + \psi(x+h,y+h) + \right. \\ \left. + \psi(x-h,y+h) + \psi(x-h,y-h) + \psi(x+h,y-h) - 6h^2 f(x,y) - \frac{h^4}{2} \nabla^2 f(x,y) \right].$$

The method most heralded by modern authorities for solving equations of this type is the method of iteration. Scarborough says this: "The greatest drawback to the method of iteration is its great length. . . Computational errors in the method of iteration are immediately evident and are self-correcting. . . The iteration process is slow, sure, and frequently long. . . The iteration process can be performed mechanically by an automatic sequence-controlled calculating machine. . . Because of the perfectly arbitrary manner in which the relaxations are made, the relaxation process cannot be carried out by an automatic calculating machine. . . If an automatic sequence - controlled calculating machine is available, a process of iteration would be used."

We start out with a coarse net (large value of h). Then when iteration given no further improvement in the ψ 's, the whole process is repeated with a finer net (smaller value of h) and the iteration is carried on until no change occurs in the ψ 's. The method of iteration starts with the upper left-hand corner of the network and proceeds to correct all network values by means of the formula presented above, using the latest computed values available $f(x,y)$, $\nabla^2 f(x,y)$, and h we know. ψ 's from a previous time step should prove a good initial guess. The process is carried out in a systematic and definite order by going from left to right until the end of a line is reached and then dropping down to the next line, just as in reading the consecutive lines of a printed page. This method of correcting the network values is continued until no further improvements can be effected by the iteration process.

Finally, I shall present in modified form a simple formula set forth by Scarborough, which enables us to see that to know merely the order of a truncation error is good information. If our error is of the order of h^n , we have

$$E = Ch^n$$

where E is our error and C is a constant of proportionality. Then for any two values h_1 and h_2 of h , the corresponding errors are

$$E_1 = Ch_1^n, E_2 = Ch_2^n$$

from which

$$\frac{E_2}{E_1} = \frac{h_2^n}{h_1^n} \quad \text{or} \quad E_2 = \left(\frac{h_2}{h_1} \right)^n E_1.$$

If $h_2 = \frac{1}{2} h_1$, then

$$E_2 = \left(\frac{1}{2}\right)^n E_1$$

Let a_1 and a_2 denote the final approximate values of the function ψ at any interior mesh point, corresponding to h_1 and h_2 respectively. Then

$$\psi = a_1 + E_1, \quad \psi = a_2 + E_2$$

Eliminating ψ and taking account of $E_2 = \left(\frac{1}{2}\right)^n E_1$, we get

$$E_2 = \frac{a_2 - a_1}{2^n - 1}.$$

This formula gives the approximate value of the inherent error at each intersection point of the network after two values of h have been used, the second value of h being half the first value.

Since $\psi = a_2 + E_2$, we can substitute the value of E_2 and get

$$\psi = a_2 + \frac{(a_2 - a_1)}{2^n - 1}$$

which gives a close approximation to the true value of ψ at any net point.

* * * * *

This is a modest paper - far too modest for one to present as representative of the magnitude of the problem involved with any degree of pride. But if you have been made anxious to any degree about an error problem or if a single question has arisen in your mind concerning the material that is presented herein (great wonder it is not ignored), then the failures this effort is bound to experience will not have been total.

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Below is a short list of books and articles used to make this paper possible. Indebtedness to these authors as well as to the courteous and obliging library staff is unbounded.

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Dorothy E. Hoover
January 24, 1958

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A short list of books and articles dealing with topics of errors and numerical analysis is furnished below in order to provide additional methods and illustrations. This list makes no pretense at completeness, but it may give lead to other items of interest to the reader in that it indicates recent key publications containing references which will enable the reader to trace the previous literature of a particular branch of numerical analysis.

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D.E.H.

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